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Bounce-Averaged Synchrotron Loss in a Dipole Field

Space Sciences Laboratory
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The Aerospace Corporation
El Segundo, Calif. 90245

18 February 1977

Interim Report

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FOR THE COMMANDER

Arturo G. Fernández
2nd Lt., U.S. Air Force
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Accurate approximate expressions are derived (in closed form) for the bounce-averaged loss rates $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$ due to synchrotron radiation in a dipole field, where $\gamma - 1$ is a particle's kinetic energy (in units of rest energy) and y is the sine of its equatorial pitch angle. The expressions for $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$ factor so that $\langle\dot{\gamma}\rangle$ is equal to $(\gamma^2 - 1)$ times a ratio of polynomials in $y^{1/4}$, while $\langle\dot{y}\rangle$ is equal to $(1/\gamma)$ times a different ratio of polynomials in $y^{1/4}$. The expressions apply to all values of γ and y for which the adiabatic theory of charged-particle motion is a good approximation.

y to the 1/4 power

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PREFACE

The author thanks Prof. John M. Cornwall for a discussion affirming the validity of (7).

CONTENTS

INTRODUCTION

Synchrotron radiation is the major deceleration mechanism for relativistic electrons trapped in Jupiter's magnetic field [Birmingham et al., 1974; Coroniti, 1974]. Since the deceleration time greatly exceeds the bounce time for such electrons, it is of great interest to have available some simple (closed-form) expressions for the corresponding bounce-averaged transport coefficients that appear in the Fokker-Planck equation. Previous investigators have considered only those electrons that mirror at or near the magnetic equator in order to simplify the averaging procedure. The purpose of the present work is to extend the averages to arbitrary equatorial pitch angles, i.e., to arbitrary mirror latitudes.

The present method of averaging makes use of a recent and excellent approximation [Davidson, 1976] for the function

$$\begin{aligned} T(y) &\equiv (1/4La) \oint [1 - y^2(B/B_0)]^{-1/2}\, ds \\ &\approx T(0) - [T(0) - T(1)]y^{3/4} \\ &\approx 1.380173 - 0.639693\, y^{3/4} \qquad (1) \end{aligned}$$

to which the particle's full bounce period $2\pi/\Omega_2 \equiv (4La/\beta c)T(y)$ is directly proportional. The equation of a dipolar field line, identified by the dimensionless label L, is $r = La \sin^2\theta$, where r is the radial coordinate, a is the planetary radius, and θ is the magnetic colatitude. The coordinate s measures arc length along the field

line. The magnetic field $\underset{\sim}{B}$ varies in magnitude from B_0 at the equator of the field line to infinity at $r = 0$. However, a particle of equatorial pitch angle $\alpha_0 \equiv \sin^{-1} y$ must mirror at field intensity $B_m \equiv B_0/y^2$ and thus remain trapped between the latitudes at which $B = B_m$. For a particle of local velocity $\underset{\sim}{\beta} c$ (where c is the speed of light) the full bounce period (interval between consecutive visits to the same mirror latitude) is as given above.

The rate of change of kinetic energy during synchrotron radiation can be written as $\dot{\gamma} m_0 c^2$, where γ is the ratio of relativistic mass (m) to rest mass (m_0). The rate of change of the sine of the equatorial pitch angle is $\dot{y}$, and its bounce average is denoted $\langle \dot{y} \rangle$. More generally, the bounce average of any quantity Q is given by

$$\langle Q \rangle \equiv \oint Q [1 - y^2 (B/B_0)]^{-1/2} \, ds \div \oint [1 - y^2 (B/B_0)]^{-1/2} \, ds. \tag{2}$$

Although $\langle \dot{\gamma} \rangle$ and $\langle \dot{y} \rangle$ suffice to describe the synchrotron "transport" of radiation-belt electrons, it also proves useful (for reasons having to do with the canonical Hamiltonian formalism) to have available expressions for the bounce-averaged time derivatives of the first two adiabatic invariants M and J.

It develops (see below) that all four of the desired bounce averages (namely $\langle \dot{\gamma} \rangle$, $\langle \dot{y} \rangle$, $\langle \dot{M} \rangle$, and $\langle \dot{J} \rangle$) can be expressed in terms of the bounce averages $\langle (B/B_0)^2 \rangle$ and $\langle (B/B_0)^3 \rangle$. Moreover,

the bounce average of $(B/B_0)^n$ is analytically related to T(y), i.e., can be derived from T(y) by taking integrals or derivatives, if n is a positive integer. Thus, the excellent approximation for T(y) given by Davidson [1976] can be manipulated to yield equally excellent approximations for the Fokker-Planck transport coefficients associated with synchrotron loss in a dipole field. This is the objective that is achieved in the present work.

SYNCHROTRON LOSS

It is well known [e.g., Jackson, 1962] that the rate of energy loss by a particle of charge q is given by

$$\dot{\gamma} m_0 c^2 = -(2q^2\gamma^6/3c)[(\dot{\boldsymbol{\beta}})^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2] \tag{3}$$

in the presence of forces that tend to change its velocity $\boldsymbol{\beta}c$. The force relevant for synchrotron radiation is that which yields $\dot{\boldsymbol{\beta}} = \boldsymbol{\Omega}_1 \times \boldsymbol{\beta}$, where $\boldsymbol{\Omega}_1 = -q\mathbf{B}/\gamma m_0 c$ is 2π times the gyrofrequency. Thus, it follows [Coroniti, 1974] from (3) that

$$\begin{aligned} \dot{\gamma} &= -(2q^4B^2/3m_0^3c^5)(\gamma^2-1)\sin^2\alpha \\ &= -(2q^4B_0^2/3m_0^3c^5)(\gamma^2-1)(B/B_0)^3y^2, \end{aligned} \tag{4}$$

where α is the local pitch angle, i.e., where $\sin^2\alpha = y^2(B/B_0)$ in the course of adiabatic charged-particle motion. It follows at once from (4) that

$$\langle\dot{\gamma}\rangle = -(2q^4 B_0^2/3m_0^3c^5)(\gamma^2 - 1)\langle(B/B_0)^3\rangle\gamma^2, \quad (5)$$

where $\langle(B/B_0)^3\rangle$ is the bounce average (to be evaluated below) of the quantity $(B/B_0)^3$.

The change in energy described by (4) is accompanied by a deflection of the local pitch angle α. In order to evaluate this, it proves convenient to decompose the particle momentum $\underset{\sim}{p} \equiv \gamma m_0 \underset{\sim}{\beta} c$ into components $p_\parallel = \underset{\sim}{p} \cdot \hat{\underset{\sim}{B}} = p \cos\alpha$ and $p_\perp = |\underset{\sim}{p} \times \hat{\underset{\sim}{B}}| = p \sin\alpha$. It is well known that

$$\gamma^2 = 1 + (p/m_0c)^2 = 1 + (p_\parallel/m_0c)^2 + (p_\perp/m_0c)^2. \quad (6)$$

However, the operative contributions of $\dot{p}_\parallel$ and $\dot{p}_\perp$ to $\dot{\gamma}$ remain to be determined. It seems evident that synchrotron radiation must leave $\beta c \cos\alpha$ unaltered, since a Lorentz transformation at this velocity along $\underset{\sim}{B}$ will bring the observer into a frame in which the electron gyrates about the field in a collapsing circle (rather than in a collapsing helix). The synchrotron radiation in this frame is symmetrical with respect to the plane of gyration, and so there is no radiation force that would impel the particle to acquire a velocity component parallel to $\underset{\sim}{B}$ in this frame. It follows from this argument that $\dot{p}_\parallel = \dot{\gamma} m_0 \beta c \cos\alpha$ in the original frame of reference.

The foregoing conclusion about $\dot{p}_\parallel$ is contrary to that assumed by Coroniti [1974], who took $\dot{p}_\parallel = 0$ without detailed justification. Thus, the present decomposition of (6) to yield

$$p_\perp \dot{p}_\perp = m_0^2 c^2 \gamma\dot{\gamma}[1 - \beta^2 \cos^2\alpha] \quad (7)$$

differs from his by the factor $[1 - \beta^2 \cos^2\alpha]$. The interpretation of the present result is that the synchrotron radiation from a relativistic electron is somewhat biased toward the direction of $p_\parallel$, and that the resulting reaction force (in the original frame of reference) is just sufficient to keep constant the component of particle velocity (not momentum) parallel to $\underset{\sim}{B}$.

By combining (7) with the foregoing relationships $\dot{p}_\parallel = \dot{\gamma} m_0 \beta c \cos\alpha$ and $y = (B_0/B)^{1/2} \sin\alpha = (B_0/B)^{1/2}(p_\perp/p)$, one can derive the expression

$$\begin{aligned} \dot{y} &= (B_0/B)^{1/2}[(\dot{p}_\perp/p) - (p_\perp/p^2)\dot{p}] \\ &= (\dot{\gamma}/y\gamma)(\gamma^2 - 1)^{-1}[1 - y^2(B/B_0)](B_0/B) \end{aligned} \tag{8}$$

for the rate of change of $\sin\alpha_0$. Thus, it follows from (4) that

$$\begin{aligned} \langle \dot{y} \rangle &= -(2q^4 y B_0^2/3\gamma m_0^3 c^5) \\ &\qquad \times [\langle (B/B_0)^2 \rangle - y^2 \langle (B/B_0)^3 \rangle] \end{aligned} \tag{9}$$

in the course of synchrotron radiation.

Since $M = p_\perp^2/2m_0 B$, it follows from (7) and (4) that the bounce-averaged time derivative of the first adiabatic invariant is given directly by

$$\begin{aligned} \langle \dot{M} \rangle &= \langle (p_\perp \dot{p}_\perp/m_0 B) \rangle = \langle (\dot{\gamma} m_0 c^2/\gamma B)(1 + \gamma^2\beta^2 \sin^2\alpha) \rangle \\ &= -(4Mq^4 B_0^2/3\gamma m_0^3 c^5)[\langle (B/B_0)^2 \rangle + (2MB_0/m_0 c^2)\langle (B/B_0)^3 \rangle]. \end{aligned} \tag{10}$$

where $1 - \beta^2 = \gamma^{-2}$ as usual. The same result can be obtained from (5) and (9) by setting $M = (\gamma^2 - 1)(m_0 c^2 y^2 / 2 B_0)$ and applying the chain rule of differential calculus. This latter (indirect) method enables one to obtain $\langle \dot{J} \rangle$ from the expression $J = 2Lap\,Y(y)$ for the second adiabatic invariant in a dipole field, where [Schulz, 1971; Davidson, 1976]

$$
\begin{aligned}
Y(y) &= 2y \int_y^1 (y')^{-2}\, T(y')\, dy' \\
&\approx 2T(0) + [6T(0) - 8T(1)]y - 8[T(0) - T(1)]y^{3/4} \\
&\approx 2.760346 + 2.357194y - 5.117544y^{3/4}. \qquad (11)
\end{aligned}
$$

It follows from the first line of (11) that $y\,Y'(y) = Y(y) - 2\,T(y)$. Thus, the second adiabatic invariant has a bounce-averaged time derivative given by

$$
\begin{aligned}
\langle \dot{J} \rangle &= 2La\left\{(m_0 c/\beta)\langle \dot{\gamma} \rangle Y(y) - (p/y)\langle \dot{y} \rangle [\,Y(y) - 2T(y)]\right\} \\
&= -\,4Lap\,(q^4 B_0^2 / 3\gamma m_0^3 c^5)\left\{[\,Y(y) - 2T(y)]\langle (B/B_0)^2 \rangle \right. \\
&\qquad \left. + \, y^2[\,(\gamma^2 - 1)\,Y(y) + 2T(y)]\langle (B/B_0)^3 \rangle \right\} \qquad (12)
\end{aligned}
$$

in closed form. It remains to be shown that the bounce averages $\langle (B/B_0)^2 \rangle$ and $\langle (B/B_0)^3 \rangle$ can be derived analytically from (1).

BOUNCE AVERAGES

It proves convenient to introduce the auxiliary variable $u = y^2$. Bounce averages of the required sort can then be obtained from the realization that

$$\frac{d}{du}\oint [1 - y^2(B/B_0)]^{1/2}\, ds$$

$$= -\frac{1}{2}\oint (B/B_0)[1 - y^2(B/B_0)]^{-1/2}\, ds$$

$$= -\frac{1}{2}\langle (B/B_0)\rangle [4La\, T(y)], \tag{13a}$$

$$\frac{d^2}{du^2}\oint [1 - y^2(B/B_0)]^{3/2}\, ds$$

$$= \frac{3}{4}\langle (B/B_0)^2\rangle [4La\, T(y)], \tag{13b}$$

$$\frac{d^3}{du^3}\oint [1 - y^2(B/B_0)]^{5/2}\, ds$$

$$= -\frac{15}{8}\langle (B/B_0)^3\rangle [4La\, T(y)], \tag{13c}$$

and (more generally)

$$\frac{d^n}{du^n}\oint [1 - y^2(B/B_0)]^{(2n-1)/2}\, ds$$

$$= (-1/2)^n (2n-1)!!\,\langle (B/B_0)^n\rangle [4La\, T(y)]. \tag{13d}$$

Thus, the problem at hand will be solved by the discovery of a prescription for evaluating the integral on the left-hand side of (13d).

The prescription follows from the fact that

$$\frac{d}{du}\left\{ y^{1-2n} \oint [1 - y^2(B/B_0]^{(2n-1)/2}\, ds \right\}$$

$$= \frac{d}{du} \oint [(1/u) - (B/B_0)]^{(2n-1)/2}\, ds$$

$$= -\frac{2n-1}{2y^{2n+1}} \oint [1 - y^2(B/B_0)]^{(2n-3)/2}\, ds. \qquad (14)$$

Since the integrals found on the left-hand side of (13d) vanish at $y = 1$ for positive integers n, it follows from (14) that

$$\oint [1 - y^2(B/B_0)]^{(2n-1)/2}\, ds$$

$$= y^{2n-1} \int_y^1 \frac{2n-1}{(y')^{2n}} \oint [1 - (y')^2(B/B_0)]^{(2n-3)/2}\, ds\, dy' \qquad (15)$$

for $n \geq 1$. Thus, it follows from (1) that

$$\oint [1 - y^2(B/B_0)]^{1/2}\, ds = y\int_y^1 (y')^{-2}[4La\, T(y')]\, dy'$$

$$\equiv 2La\, Y(y) \qquad (16a)$$

for $n = 1$, from (11) that

$$\oint [1 - y^2(B/B_0)]^{3/2}\, ds = 3y^3 \int_y^1 (y')^{-4}[2La\, Y(y')]\, dy'$$

$$\approx (2La/3)\{6T(0) + 9[3T(0) - 4T(1)]y$$

$$- [T(0) - 4T(1)]y^3 - 32[T(0) - T(1)]y^{3/4}\} \qquad (16b)$$

for $n = 2$, and from (16b) that

$$\oint [1 - y^2(B/B_0)]^{5/2}\, ds$$

$$\approx 4La\{T(0) + (15/8)[3T(0) - 4T(1)]y$$

$$- (5/12)[T(0) - 4T(1)]y^3$$

$$- (320/51)[T(0) - T(1)]y^{3/4}$$

$$+ (3/136)[3T(0) - 20T(1)]y^5\} \qquad (16c)$$

for $n = 3$. It follows from (13) and (16), upon insertion of the auxiliary variable $u = y^2$, that

$$\langle(B/B_0)\rangle T(y) \approx 3[T(0) - T(1)]y^{-5/4} - [3T(0) - 4T(1)]y^{-1}, \qquad (17a)$$

$$6\langle(B/B_0)^2\rangle T(y) \approx$$

$$10[T(0) - T(1)]y^{-13/4} - 3[3T(0) - 4T(1)]y^{-3}$$

$$+ [4T(1) - T(0)]y^{-1}, \qquad (17b)$$

and

$$408\langle(B/B_0)^3\rangle T(y) \approx$$

$$520[T(0) - T(1)]y^{-21/4} - 153[3T(0) - 4T(1)]y^{-5}$$

$$+ 34[4T(1) - T(0)]y^{-3} + 9[20T(1) - 3T(0)]y^{-1}. \qquad (17c)$$

Evaluation of the relevant transport coefficients is achieved by substituting (17) in (5), (9), (10), and (12). One obtains, for

example

$$\langle\dot{\gamma}\rangle \approx -(q^4B_0^2/612m_0^3c^5)(\gamma^2-1)\{520[T(0)-T(1)]y^{-13/4}$$
$$-153[3T(0)-4T(1)]y^{-3}+34[4T(1)-T(0)]y^{-1}$$
$$+9[20T(1)-3T(0)]y\} \div T(y) \qquad (18)$$

upon substitution of (17c) in (5). Since $T(0) \sim 2T(1)$, all of the square-bracketed coefficients in (18) are positive. The corresponding expression for $\langle\dot{y}\rangle$ is given by

$$\langle\dot{y}\rangle \approx -(q^4B_0^2/612\gamma m_0^3c^5)\{160[T(0)-T(1)]y^{-9/4}$$
$$-51[3T(0)-4T(1)]y^{-2}+34[4T(1)-T(0)]$$
$$-9[20T(1)-3T(0)]y^2\} \div T(y) \qquad (19)$$

upon substitution of (17b) and (17c) in (9). It is easy to verify by direct evaluation of (19) and (18) that $\langle\dot{y}\rangle \to 0$ and that $\langle\dot{\gamma}\rangle$ approaches the equatorial value of (4) as $y \to 1$. These desired results are assured by the fact that $\langle(B/B_0)^3\rangle \to \langle(B/B_0)^2\rangle \to \langle(B/B_0)\rangle \to 1$ when this limit is taken in (17). The bounce average of B/B_0, as given by (17a), is not actually utilized in the theory of synchrotron loss but is included here in the interest of algebraic completeness.

It is convenient that the y dependence and the γ dependence appear in separate factors in both (18) and (19). This factorization makes the quantities $(-3m_0^3c^5/2q^4B_0^2)(m_0c/p)^2\langle\dot{\gamma}\rangle$ and $(-3m_0^3c^5/2q^4B_0^2)\gamma\langle\dot{y}\rangle$ universal functions of y. Both are plotted in Figure 1, the former as a solid curve and the latter as a dashed curve. Thus, the solid curve represents $y^2\langle(B/B_0)^3\rangle$ and the

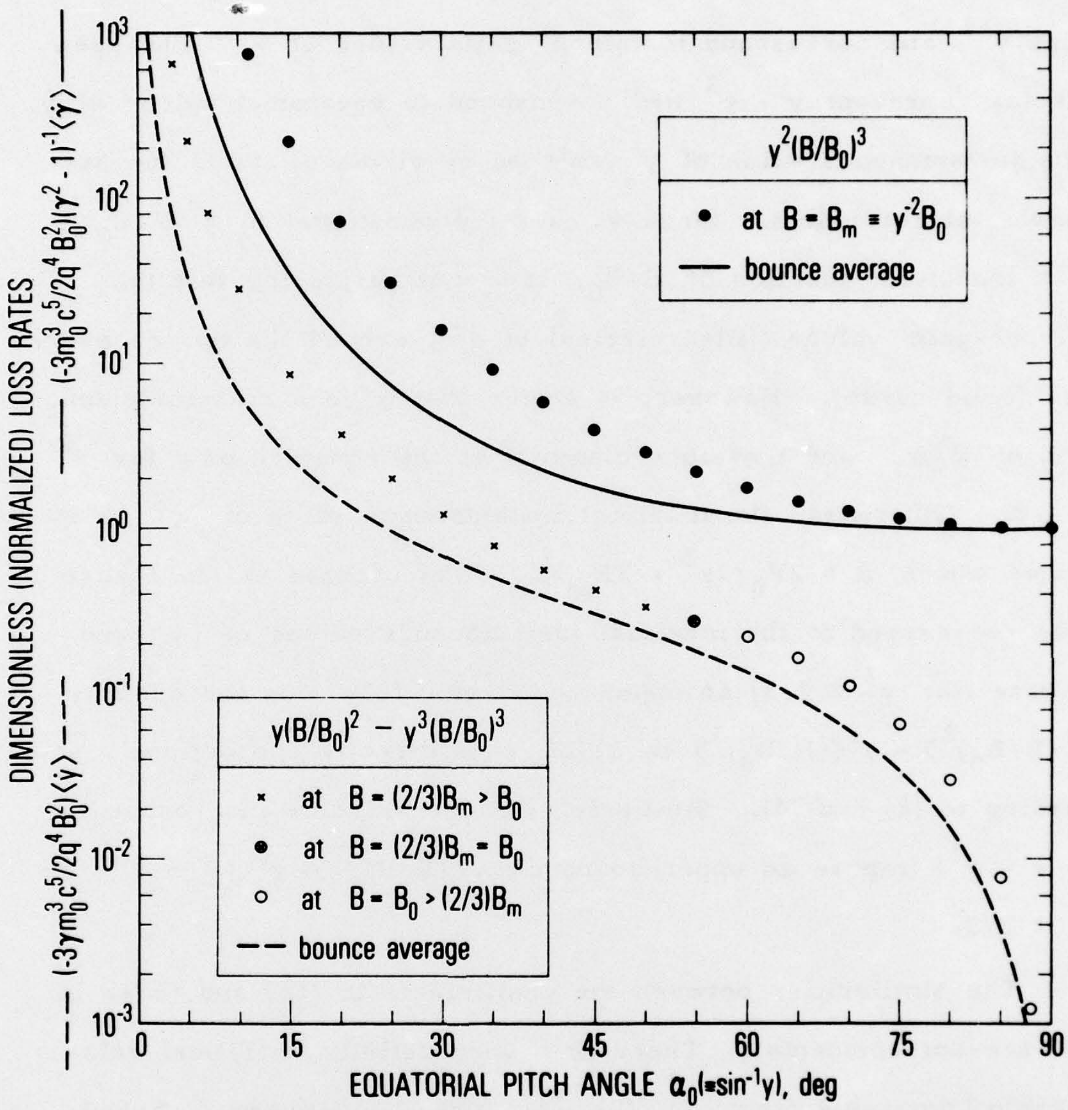

Figure 1. Maximal values ("data" points) and bounce averages (dashed and solid curves, respectively) of the quantities $y(B/B_0)^2 - y^3(B/B_0)^3$ and $y^2(B/B_0)^3$, which are proportional (respectively) to the loss rates $\dot{y}$ and $\dot{\gamma}$ due to synchrotron radiation by particles trapped in a dipolar magnetic field

dashed curve represents $y\langle(B/B_0)^2\rangle - y^3\langle(B/B_0)^3\rangle$. The isolated "data" points represent instantaneous values of $y^2(B/B_0)^3$ and $y(B/B_0)^2 - y^3(B/B_0)^3$, i.e., correspond to instantaneous values of $\dot{\gamma}$ and $\dot{y}$ plotted on the same scale. The filled circles represent y^{-4} and correspond to mirror-point values of $\dot{\gamma}$. The open circles represent $y - y^3$ and correspond to equatorial values of $\dot{y}$. The instantaneous value of $\dot{y}$ vanishes by virtue of (8) at the particle's mirror points. Since $\dot{\gamma}$, being proportional to $y^2(B/B_0)^3$, is a monotonic function of B/B_0, it is not surprising that the mirror-point values (filled circles) of $|\dot{\gamma}|$ exceed the bounce average (solid curve). However, it seems that $|\dot{y}|$ is a monotonic function of B/B_0, and therefore maximal at the equator, only for $y^2 \geq 2/3$. Otherwise, the maximal instantaneous value of $|\dot{y}|$ is attained where $B = 2B_0/3y^2 = 2B_m/3$. The crosses (×) in Figure 1 thus correspond to the maximal instantaneous values of $|\dot{y}|$ and impose (for $y^2 \leq 2/3$) an upper bound of $(4/27y^3)$ on the quantity $y\langle(B/B_0)^2\rangle - y^3\langle(B/B_0)^3\rangle$ to which $\dot{y}$ is directly proportional, according to (8) and (4). Similarly, the open circles (corresponding to $y - y^3$) impose an upper bound on $y\langle(B/B_0)^2\rangle - y^3\langle(B/B_0)^3\rangle$ for $y^2 \geq 2/3$.

The similarities between the coefficients in (18) and those in (19) are not accidental. There is a very definite analytical relationship [derivable from (1), (2), (5), and (9)] between $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$. This relationship follows from the property that

$$(1/y)\langle\dot{\gamma}\rangle T(y) = -(\gamma^2 - 1)(q^4 B_0^2/12 m_0^3 c^5 La) \times y \oint (B/B_0)^3 [1 - y^2(B/B_0)]^{-1/2}\, ds, \quad (20a)$$

whereas

$$(1/y)\langle\dot{y}\rangle T(y) = -(1/\gamma)(q^4 B_0^2/12 m_0^3 c^5 La) \times \oint (B/B_0)^2 [1 - y^2(B/B_0)]^{1/2}\, ds. \quad (20b)$$

Inspection now reveals that the right-hand side of (20a) is precisely equal to $-\gamma(\gamma^2 - 1)$ times the derivative (with respect to y) of the right-hand side of (20b). Thus, it follows from (20) that

$$\frac{\langle\dot{\gamma}\rangle T(y)}{y(\gamma^2 - 1)\gamma} = -\frac{d}{dy}\left[\frac{T(y)}{y}\langle\dot{y}\rangle\right] \quad (21a)$$

and from (21a) that

$$\langle\dot{y}\rangle = \frac{y}{T(y)}\int_y^1 \frac{T(y')\langle\dot{\gamma}(y')\rangle\, dy'}{y'(\gamma^2 - 1)\gamma}, \quad (21b)$$

since $\langle\dot{y}\rangle = 0$ by virtue of (8) and (9) for $y = 1$. Inspection of (18) and (19) reveals that $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$, as calculated via (17), satisfy precisely the relationship imposed by (21a). This fact tends to confirm the correctness of the algebraic manipulations carried out above.

FOKKER-PLANCK EQUATION

If processes other than synchrotron loss are neglected as a first approximation, the evolution of the bounce-averaged canonical phase-space density $\bar{f}$ is given [Haerendel, 1968; Schulz and Lan-

zerotti, 1974] by

$$\frac{\partial \bar{f}}{\partial t} + \frac{\partial}{\partial M}\left[\langle \dot{M} \rangle \bar{f}\right]_J + \frac{\partial}{\partial J}\left[\langle \dot{J} \rangle \bar{f}\right]_M = 0. \tag{22}$$

Implementation of (22), however, would require one to express $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ as functions of M and J, whereas the present work yields $\langle \dot{M} \rangle$ and $\langle \dot{J} \rangle$ mainly as functions of γ and y. It is certainly possible to express γ and y as functions of M and J [Chen and Stern, 1975] by means of accurate analytical approximations unrelated to (1). However, this would disturb the internal consistency heretofore maintained in the present work, in which all analytical approximations have been derived from (1).

The transformation of (22) to the variables γ and y, moreover, would make unnecessary any such appeal to analytical approximations not derivable from (1). All that one requires in this case is the Jacobian of the transformation from (M, J) to (γ, y). Since it has been shown above that $M = (\gamma^2 - 1)(m_0 c^2 y^2 / 2B_0)$ and $J = 2Lap\,Y(y)$, where $yY'(y) = Y(y) - 2T(y)$, it follows that the relevant Jacobian is given by

$$\partial(M, J)/\partial(\gamma, y) = -4La\gamma p(m_0 c^2 / B_0)\, y\, T(y). \tag{23}$$

Thus, it follows [Haerendel, 1968; Schulz and Lanzerotti, 1974] from (22) and (23) that

$$\frac{\partial \bar{f}}{\partial t} + \frac{1}{\gamma p}\frac{\partial}{\partial \gamma}\left[\gamma p \langle \dot{\gamma} \rangle \bar{f}\right]_y + \frac{1}{yT(y)}\frac{\partial}{\partial y}\left[y\, T(y) \langle \dot{y} \rangle \bar{f}\right]_\gamma = 0, \tag{24}$$

where $p = (\gamma^2 - 1)^{1/2} m_0 c$. The convenience of (24) is enhanced by the fact that the γ dependence and the y dependence of $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$ appear in separate factors in (18) and (19). The unit Jacobian in (22) arises from the fact that M and J are canonical action variables of the underlying phase space [Haerendel, 1968; Schulz and Lanzerotti, 1974]. However, the forms of $\langle\dot{M}\rangle$ and $\langle\dot{J}\rangle$ given by (10) and (12) are less convenient for use in (22) than are the forms of $\langle\dot{\gamma}\rangle$ and $\langle\dot{y}\rangle$ given by (18) and (19) for use in (24).

In the likely event that additional dynamical processes (besides synchrotron loss) are operative, the corresponding Fokker-Planck terms should be added to (24). These might include collisional terms leading to energy loss (additional $\langle\dot{\gamma}\rangle$), pitch-angle diffusion (D_{yy}), and energy diffusion ($D_{\gamma\gamma}$). They might also include wave-particle interaction terms requiring the specification not only of D_{yy} and $D_{\gamma\gamma}$, but also of the off-diagonal term $D_{\gamma y}$ ($= D_{y\gamma}$). Finally, they might include distributed sources (S) and (charge-exchange) sinks ($-\langle\tau_q^{-1}\rangle\bar{f}$) of trapped radiation, as well as radial diffusion (D_{LL}) at constant M and J. There is no contradiction in describing radial diffusion as occuring at constant M and J, even when the terms describing other dynamical processes in the same Fokker-Planck equation are being evaluated at constant y or γ. The same phase-space density $\bar{f}$ is simultaneously a function of (M, J, L; t) and of (γ, y, L; t). The representation chosen for the purpose of evaluating a given term in the Fokker-Planck equation should be determined by the set of variables with respect to which that individual term has been explicitly specified.

DISCUSSION

The present work has yielded expressions (in closed form) for the bounce-averaged time derivatives of a particle's kinetic energy and equatorial pitch angle (more precisely, the sine thereof). The expressions thus derived are contingent on an accurate representation for $T(y)$, as given in (1). Experience has shown [Schulz, 1971] that functions [such as $Y(y)$ in (11)] which are derived from $T(y)$ by integration are about as accurate (by percentage) as $T(y)$ itself. The present representation [Davidson, 1976] for $T(y)$ is accurate within 0.6% for $0 \le y \le 1$. Since the bounce average of $(B/B_0)^n$ is found to involve n integrations over $T(y)$ and n differentiations of the result, the accumulation of additional error is possible but not likely. Since the bounce averages $\langle (B/B_0)^2 \rangle$ and $\langle (B/B_0)^3 \rangle$ are the essential ingredients in $\langle \dot{\gamma} \rangle$ and $\langle \dot{y} \rangle$, it is likely that $\langle \dot{\gamma} \rangle$ and $\langle \dot{y} \rangle$ are thereby represented about as accurately as $T(y)$ itself, i.e., within $\sim 0.6\%$ for $0 \le y \le 1$.

Moreover, the present expressions for $\langle \dot{\gamma} \rangle$ and $\langle \dot{y} \rangle$ are comparatively simple and easy to program. Each is a ratio of functions of $y^{1/4}$, having a numerator of four terms and a denominator of two terms. The superficially cumbersome numerical coefficients involve only $T(0)$ and $T(1)$ in linear combination, where $T(0) \approx 1.3801730$ and $T(1) \approx 0.7404805$. Thus, the expressions for $\langle \dot{\gamma} \rangle$ and $\langle \dot{y} \rangle$ given respectively by (18) and (19) should be fully satisfactory for most analytical purposes.

The component of acceleration $(\dot{\underset{\sim}{\beta}} c)$ associated with curvature

of the field line $(\partial \hat{\underset{\sim}{B}}/\partial s)$ has been neglected here, as in Coroniti [1974], in its contribution to $\dot{\gamma}$ and $\dot{y}$. This approximation should be valid for particles having $|(pc/qB_0)(\partial\hat{\underset{\sim}{B}}/\partial s)_0| \ll y$, i.e., for particles to which the laws of adiabatic motion (also tacitly assumed here) apply.

The present method for calculating bounce averages of the form $\langle (B/B_0)^n \rangle$ should be useful in many applications besides the theory of synchrotron loss in a dipole field. The method lends itself readily, for example, to the calculation of collisional transport coefficients in the presence of a scattering medium distributed with a density proportional to $(B/B_0)^n$ along a field line. Although the method presently requires n to be a positive integer, it should be easy to find an analogous method for handling negative integers n. Finally, field geometries other than the dipole could be treated by introducing the forms of $T(y)$ appropriate to the other field geometries of interest.

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